Gaussian Elimination

Gaussian elimination is a systematic solution for solving a linear system of equations and can therefore form a numerical method[1]. It is based on the method of 'planned' elimination method that is used for solving simultaneous equations[2] and is generally applied to the matrix-vector[3] form of the linear system:

$$A\underline{x} = \underline{b},$$

where A is an $n \times n$ matrix and $\underline{x}$ and $\underline{b}$ are n-vectors.

Solving a system of two simultaneous equations by Gaussian elimination

Consider the example[2]

$$2x + y = 7 \qquad (1)$$
$$3x + 2y = 12 \qquad (2)$$

The simplest elimination perhaps is to eliminate y by either doubling equation (1) or halving equation (2) and subtracting. However in Gaussian elimination (unless problems arise) the variables are eliminated in order, from each equation in turn, so in this case x should be eliminated from equation (2). In Gaussian elimination, this can be done by multiplying the second equation by $2/3$ to arrive at equation (2a), so that the coefficient in x is the same for both equation, and subtracting one equation from the other. Multiplying equation (2) by

$$2x + y = 7 \qquad (1)$$
$$2x + \frac{4}{3}y = 8 \qquad (2a)$$

Subtracting equation (1) from equation (2a) gives

$$\frac{1}{3}y = 1 \qquad (2b)$$

and, in Gaussian elimination, this equation replaces (2a) in the system of equations, so that the system becomes

$$2x + y = 7 \qquad (1)$$
$$\frac{1}{3}y = 1 \qquad (2b).$$

For this and larger $(n \times n)$ systems, the final equation has just one variable, and can therefore can easily be solved to find its value. In the example, equation (2b) gives

$$y = 3.$$

Once this value is known the equation above with two variables can easily be solved. In the example equation, the value $y = 3$ is substituted into equation (1) to give

$$2x + 3 = 7$$

[1] Numerical Methods for solving linear systems of equations
[2] Solving a pair of simultaneous equations by elimination
[3] Matrix Definitions

and hence that $x = 2$. The method of evaluating the final variable from the final equation and then substituting that value in the penultimate equation to find the penultimate value and so on until all the values have been found is called *back substitution.*

Solving a 2×2 matrix-vector problem by Gaussian elimination

Gaussian elimination is usually applied to the matrix-vector form of a linear system of equations. The example above can be written in the form:

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}.$$

The method of Gaussian elimination is the same as before and it is to systematically carry out row operations so that the matrix is *upper triangular*, that is that every element below the diagonal is zero. In this case only the '3' needs to be changed to a zero through a row operation.

Because row operations are applied to the matrix and the vector together then the system above is often written in the form of an *augmented matrix.* For this example the augmented matrix for this example is as follows

$$\begin{pmatrix} 2 & 1 & 7 \\ 3 & 2 & 12 \end{pmatrix}.$$

In order to change the '3' to a '0' by row operations, the second row is replaced by '$\frac{2}{3} \times$ row 2 minus row 1' to give

$$\begin{pmatrix} 2 & 1 & 7 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

Back substitution is then applied; the second row effectively states that $\frac{1}{3}y = 1$, giving $y = 3$ and then this value is substituted into the first equation to give $x = 2$.

For a 3x3 matrix, the Gaussian elimination method is outlined and demonstrated on a Spreadsheet in a separate document[4].

Solving a general matrix-vector equation by Gaussian elimination

For the solution of a general matrix-vector equation $A\underline{x} = \underline{b}$ by Gaussian elimination, the method is applied as above. We only need to extend the systematic approach to the larger systems. Let us consider the general matrix-vector system in more detail;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & & \cdots & a_{2n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & & a_{n,n-1} & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}.$$

[4] Gaussian Elimination for a 3x3 System

The initial goal of Gaussian elimination is to change the matrix into an upper triangular matrix by using row operations. The normal procedure is to move through the successive columns of the matrix, making the sub-diagonal elements zero.

Starting with the first column and the second row element a_{21}, the following method changes the element to zero

$$a_{21} \leftarrow a_{21} - (\frac{a_{21}}{a_{11}})\, a_{11}\ .$$

The same operation needs to be applied to the whole row and the vector $\underline{b}$;

$$a_{2k} \leftarrow a_{2k} - \left(\frac{a_{21}}{a_{11}}\right) a_{1k} \text{ and } b_k \leftarrow b_k - \left(\frac{a_{21}}{a_{11}}\right) b_k \text{ for } k = 1..n\ .$$

Moving on to the first column and third row element, the operation

$$a_{31} \leftarrow a_{31} - (\frac{a_{31}}{a_{11}})\, a_{11}$$

changes the element to zero, and again the same operation must be applied to the row and to the vector $\underline{b}$

$$a_{3k} \leftarrow a_{3k} - \left(\frac{a_{31}}{a_{11}}\right) a_{1k} \text{ for } k = 1..n\ .$$

This operation is carried out on every row j (j=1..n) in the first column;

$$a_{jk} \leftarrow a_{jk} - \left(\frac{a_{j1}}{a_{11}}\right) a_{1k} \text{ for } j = 2..n\ .$$

For each column in turn, the sub-diagonal elements are eliminated in the same way, for example for the i^{th} column

$$a_{jk} \leftarrow a_{jk} - \left(\frac{a_{ji}}{a_{ii}}\right) a_{ik} \text{ for } i = 1..n-1, \qquad j = i..n, \qquad k = i..n\ .$$

Once this process is completed, the matrix is upper triangular;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & & \cdots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & & \cdots & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & & . & . \\ 0 & 0 & 0 & & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & & \cdots & 0 & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix},$$

but note that the a_{ij} and b_j values will have generally been altered.

Yje process of back substitution can now commence. Starting with the equation on the final row:

$$a_{nn} x_n = b_n$$

which has the solution $x_n = b_{nn}/a_n$.

Moving backwards through the equations, the penultimate row may be written as

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}\,x_n = b_{n-1}$$

and hence

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

This process is continued, so that all the x_i are evaluated:

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}\,x_j}{a_{ii}} \quad for\ i = n..1\ .$$

Pivoting

It is possible that the diagonal element a_{ii} is zero during Gaussian elimination. It may be that this is an outcome of the matrix being singular. However, it is more likely that this is a 'chance' situation and the system of equations has a unique solution.

There is also an issue in computation in that is the diagonal element a_{ii} is 'small' then this may propagate large numerical error.

In order to overcome the potential problems, '*pivoting*' is introduced. Pivoting involves investigating the rows below the diagonal at the stage of eliminating each column in the Gaussian elimination and swapping the row in which a_{ij} (*i*>*j*) is greatest with the *i*th row.

Efficiency

In computational terms the Gaussian elimination involved three loops and hence the method $O(n^3)$[5].

Condition of the matrix

The success of any method for solving a matrix-vector system is dependent on the 'condition' of the matrix. The worst case of this is that the matrix is singular and in such a case no unique solution exists (and hence any method for finding one will fail). In order to quantify the condition of matrix a measure of the condition, called the *condition number*[6] is used; a large condition number indicates the closer the matrix is to being singular and the more that errors (e.g. numerical or experimental) will be magnified.

[5] Big O notation in computing
[6] Condition Number of a Matrix